2 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours (plus 5 minutes reading) **Exam date:** 10th August, 1999

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

2 (a) Evaluate $\dfrac{2{\cdot}3}{\sqrt[3]{2{\cdot}76 - 1{\cdot}09^2}}$, correct one decimal place.

2 (b) Find the values of x for which $|2x - 1| < 5$.

2 (c) Factorize completely $2x^3 - 54$.

2 (d) Express $\dfrac{3}{2\sqrt{3} - 1}$ in the form $a + b\sqrt{3}$.

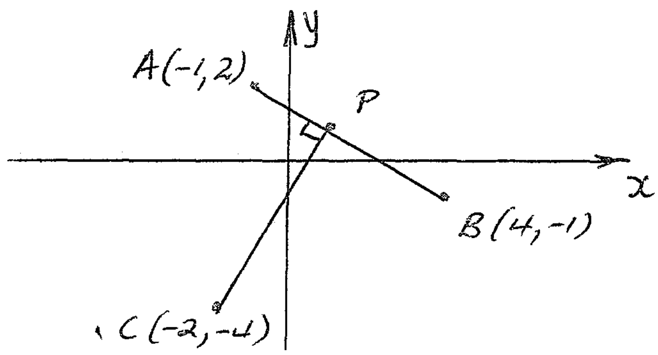
2 (e) Simplify fully $\dfrac{1 - \cos^2\theta}{\sin\theta\cos\theta}$.

2 (f) Find, to the nearest degree, the acute angle between the line $3x - 2y + 7 = 0$ and the x-axis.

QUESTION TWO (Start a new answer booklet)

Marks

9 (a)



In the diagram above, AB is the interval joining the points $A(-1, 2)$ and $B(4, -1)$. P is the foot of the perpendicular drawn from the point $C(-2, -4)$ to AB.

(i) Copy this diagram into your answer booklet.

(ii) Show that the distance from A to B is $\sqrt{34}$ units.

(iii) Find the gradient of the line AB and hence show that its equation is $3x + 5y - 7 = 0$.

(iv) Find the perpendicular distance from C to AB and hence find the area of $\triangle ABC$.

(v) Find the coordinates of the midpoint M of AC and show it on your diagram. Use this point to find the coordinates of point D so that $ABCD$ is a parallelogram.

3 (b) For the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$ find:

(i) the domain of $f(x)$,

(ii) the range of $f(x)$.

QUESTION THREE (Start a new answer booklet)

Marks

5 (a) Differentiate the following with respect to x:

(i) $x^2 - \frac{1}{x^2}$,

(ii) $x^2 e^x$ (use the product rule),

(iii) $\frac{\log_e x}{x}$ (use the quotient rule).

3 (b) The function $y = ax^3 + bx + 4$ has a stationary point at $(1, -2)$. Write down two equations and solve them to find the values of a and b.

4 (c) Find:

(i) $\int \frac{1}{(x-4)^2}\, dx$,

(ii) $\int_5^{e+4} \frac{1}{x-4}\, dx$.

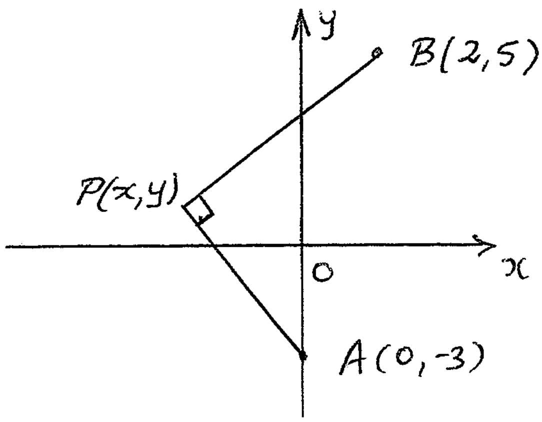
x^{-2}

$-2x^{-3}$

QUESTION FOUR (Start a new answer booklet)

Marks

4 (a)

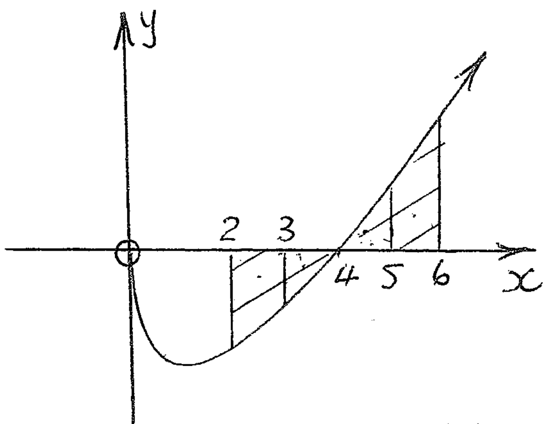


In the diagram above, points A and B have coordinates $(0, -3)$ and $(2, 5)$ respectively. $P(x, y)$ is a point such that PA is perpendicular to PB.

(i) Prove that the locus of P is the circle $x^2 + y^2 - 2x - 2y - 15 = 0$.

(ii) Find the centre and the radius of this circle.

4 (b)



The diagram above shows the graph of $y = x \log_e \left(\frac{x}{4} \right)$.

(i) Copy and complete the following table, giving your answers correct to three decimal places where necessary.

x	2	3	4	5	6
y					

(ii) By considering areas above and below the x-axis, use Simpson's rule with these five function values to evaluate the area shaded on the graph. Give your answers correct to two decimal places.

4 (c) Find the equation of the tangent to the curve $y = 1 + \cos 2x$ at the point $(\frac{\pi}{4}, 1)$. Give your answer in general form.

QUESTION FIVE (Start a new answer booklet)

Marks

6 (a) The nth term of an arithmetic series is given by $T_n = 9 - 2n$.

(i) List the first five terms and hence find the first term and the common difference.

(ii) Show that the sum S_n of the first n terms is $8n - n^2$.

(iii) Hence find the least number of terms of the series which need to be taken for this sum to be less than -945.

6 (b) A quadratic function has equation $f(x) = mx^2 - 4mx - m + 15$, where m is a constant.

Find the values of m for which:

(i) 3 is a zero of $f(x)$,

(ii) $f(x)$ is positive definite,

(iii) $\alpha + \beta = \alpha\beta$, where α and β are the zeroes of $f(x)$.

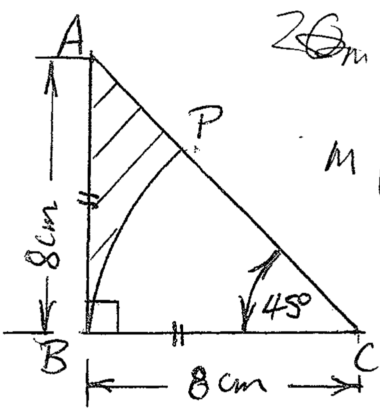
QUESTION SIX (Start a new answer booklet)

Marks

4 (a)

$4x^2 - 16x + 11$

$m(20m - 60) = 0$



In the diagram above, ABC is a right-angled isosceles triangle with $\angle ABC = 90°$ and $AB = BC = 8$ cm. Arc BP with centre C and radius CB is drawn to meet AC in P.

Find, in exact form:

(i) the area of the shaded region ABP,

(ii) the perimeter of the shaded region.

$\frac{1}{2}r^2(\theta - \sin\theta)$

8 (b) The function $y = f(x)$ is given by the equation $y = \frac{1}{3}x^3 - x^2 + 1$.

(i) Find any stationary points and determine their nature.

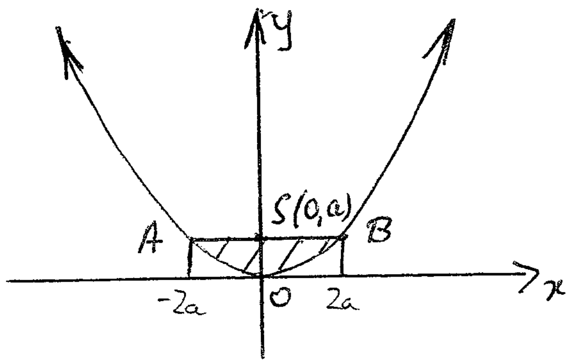
(ii) Find any points of inflexion.

(iii) Sketch $y = f(x)$ in the domain $-2 \leq x \leq 3$ giving coordinates of all turning points, points of inflexion and end-points. You need not find the x-intercepts.

QUESTION SEVEN (Start a new answer booklet)

Marks

4 (a)

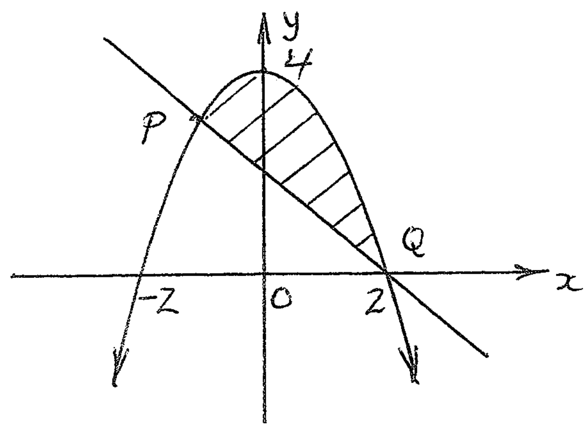


The diagram shows the graph of the parabola $x^2 = 4ay$. The interval AB is the focal chord that is parallel to the directrix and has equation $y = a$.

(i) Find the coordinates of the points A and B.

(ii) Find the area enclosed by the focal chord AB and the parabola.

4 (b)



In the diagram above, the functions $y = 4 - x^2$ and $y = 2 - x$ intersect in the points P and Q.

(i) By solving these equations simultaneously, show that the x-values at P and Q are -1 and 2 respectively.

(ii) Find the volume generated when the area enclosed by the two functions is rotated about the x-axis.

4 (c) The population of a small country town is growing at a rate that is proportional to the number of people in the town. The population P after t years is therefore $P = P_0 e^{kt}$, where k is a constant and P_0 is the initial population.

If the initial population is 6000 and ten years later the population is 9000 find:

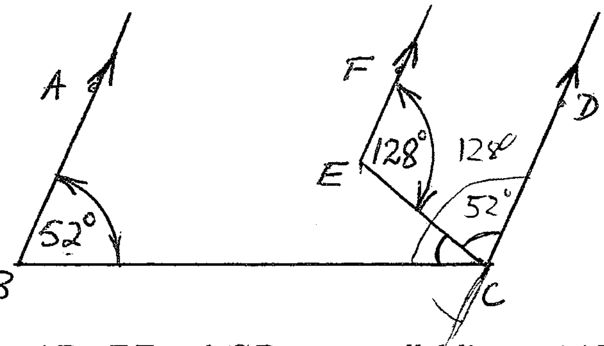
(i) the value of k in exact form,

(ii) how many years (to the nearest whole number) it will take for the population to reach five times its initial value.

QUESTION EIGHT (Start a new answer booklet)

Marks

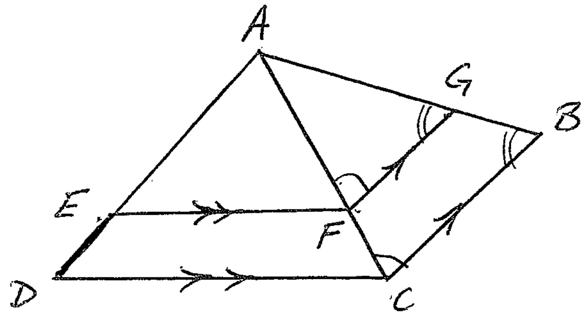
3 (a)



In the diagram above, AB, EF and CD are parallel lines. $\angle ABC = 52°$ and $\angle FEC = 128°$.

Find the size of $\angle BCE$ stating all reasons.

3 (b)



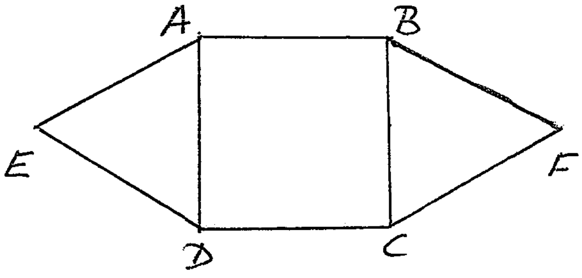
In the diagram above, EF is parallel to DC and FG is parallel to CB.

(i) Copy this diagram into your answer booklet.

(ii) If $AG = 6$ cm, $AB = 9$ cm and $AE = 8$ cm, show this information on your diagram and find, stating all reasons, the length of the interval ED.

AG : AB = AF : AC

6 (c)



In the diagram above, $ABCD$ is a square, and equilateral triangles AED and BFC have been constructed on the sides AD and BC respectively.

Copy the diagram into your answer booklet and use it to prove that:

(i) $\triangle ABF \equiv \triangle CDE$,

(ii) $AF = EC$,

(iii) $AFCE$ is a parallelogram.

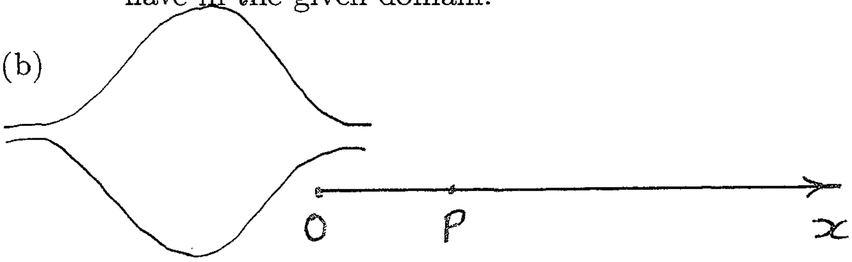
QUESTION NINE (Start a new answer booklet)

Marks

4 (a) (i) Sketch on the same number plane the graphs of $y = 3\sin 2x$ and $y = 1 - \cos x$, for $0 \leq x \leq 2\pi$.

(ii) Hence determine the number of solutions the equation $3\sin 2x + \cos x = 1$ will have in the given domain.

8 (b)



In the diagram above, the particle P is moving from rest from a fixed point O in the positive direction. The displacement x metres of the particle from O at time t seconds is given by:

$$x = 30t - 150 + 150e^{-0\cdot 2t}.$$

(i) Show that its velocity at time t seconds is:

$$v = 30(1 - e^{-0\cdot 2t}).$$

(ii) Explain why the velocity will never exceed 30 metres per second.

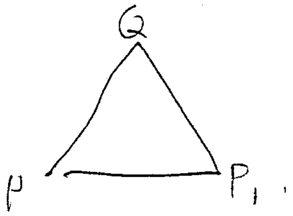
(iii) Find after what time, to the nearest 0·1 second, the particle will attain a velocity of 15 metres per second, and find its displacement, to the nearest metre, at that time.

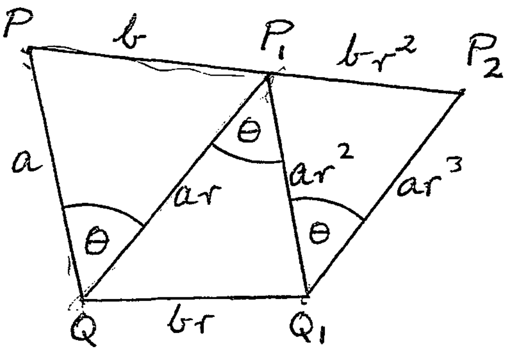
(iv) Find the acceleration of the particle at O.

QUESTION TEN (Start a new answer booklet)

Marks

[4] (a)





The diagram above shows three similar triangles:

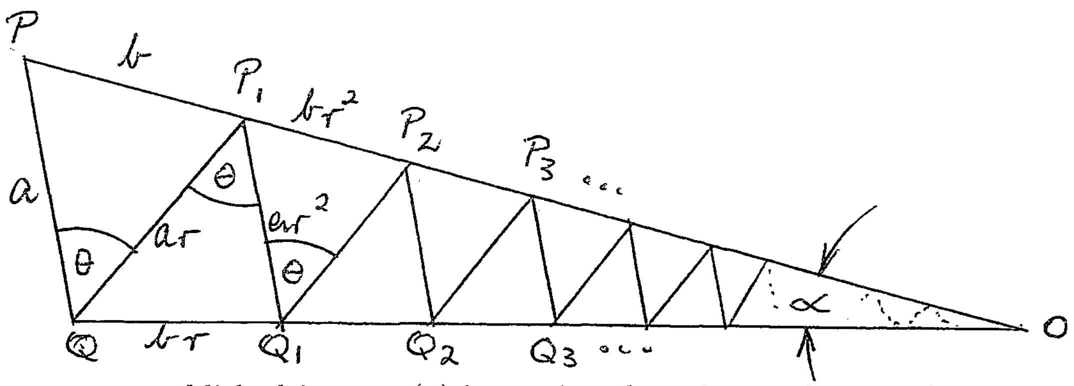
$$\triangle PQP_1 \parallel\!| \triangle QP_1Q_1 \parallel\!| \triangle P_1Q_1P_2$$

with $\angle PQP_1 = \angle QP_1Q_1 = \angle P_1Q_1P_2 = \theta$. The lengths of the sides PQ, QP_1, P_1Q_1 and Q_1P_2 are a, ar, ar^2 and ar^3 respectively and form a geometric sequence where $0 < r < 1$. Also, sides PP_1, QQ_1 and P_1P_2 have length b, br and br^2 respectively.

(i) Copy the diagram into your answer booklet and use the fact that the triangles are similar to prove that the points P, P_1 and P_2 are collinear.

(ii) Show that the area of $\triangle PQP_1$ is $\frac{1}{2}ra^2 \sin\theta$, and hence show that the ratio of the area of $\triangle QP_1Q_1$ to the area of $\triangle PQP_1$ is r^2.

[8] (b)



The pattern established in part (a) is continued as shown above to form an infinite sequence of similar triangles PQP_1, QP_1Q_1, $P_1Q_1P_2$ . . . . Let the lines PP_1P_2 . . . and QQ_1Q_2 . . . meet at O, and let $\angle QOP = \alpha$.

(i) Use the sum of an infinite sequence to find the lengths of OP and OQ.

(ii) Also using infinite sequences, show that the area of $\triangle QOP$ is $\dfrac{ra^2 \sin\theta}{2(1-r^2)}$.

(iii) Using parts (i) and (ii) above, prove that $\dfrac{\sin\alpha}{\sin\theta} = \dfrac{a^2(1-r^2)}{b^2}$.

(iv) When $\theta = 60^\circ$ and $r = 0{\cdot}9$, it can be shown by the cosine rule that $b = \dfrac{a\sqrt{91}}{10}$ (you need not prove this). Find the value of α to the nearest minute.

GJ

The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ \ n \neq -1; \ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \ \ x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \ \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \ \ a > 0, \ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \ \ x > 0$$

QUESTION ONE

(a) $\dfrac{2{\cdot}3}{\sqrt[3]{2{\cdot}76 - 1{\cdot}09^2}} = 1{\cdot}978\ldots.$

$\doteqdot 2{\cdot}0$ (to one decimal place) $\boxed{\surd\surd \text{ (2 marks correct answer)}}$

(b) $|2x - 1| < 5$.
The distance from x to $\frac{1}{2}$ is less than $\frac{5}{2}$.

So $-2 < x < 3$ $\boxed{\surd\surd\ (-1 \text{ each error})}$

(c) $2x^3 - 54 = 2(x^3 - 27)$

$= 2(x-3)(x^2 + 3x + 9)$ $\boxed{\surd\surd\ (-1 \text{ each error})}$

(d) $\dfrac{3}{2\sqrt{3}-1} = \dfrac{3}{2\sqrt{3}-1} \times \dfrac{2\sqrt{3}+1}{2\sqrt{3}+1}$ $\boxed{\surd}$

$= \dfrac{3(2\sqrt{3}+1)}{12-1}$

$= \dfrac{3}{11} + \dfrac{6\sqrt{3}}{11}.$ $\boxed{\surd}$

(e) $\dfrac{1-\cos^2\theta}{\sin\theta\cos\theta} = \dfrac{\sin^2\theta}{\sin\theta\cos\theta}$ $\boxed{\surd}$

$= \tan\theta$ $\boxed{\surd}$

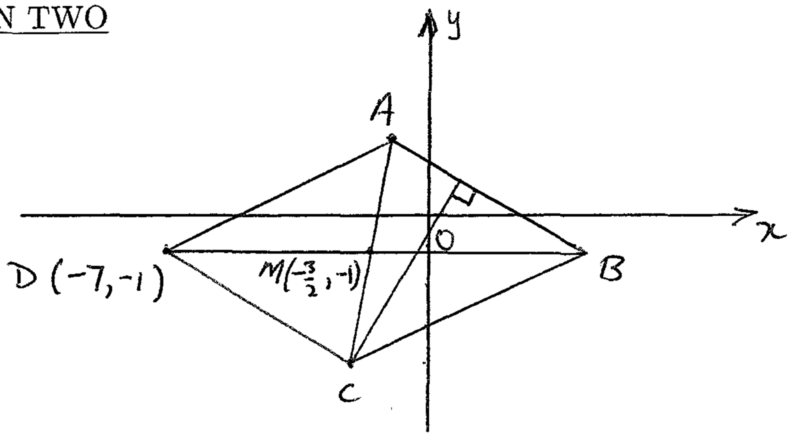
(f) $3x - 2y + 7 = 0$

$\tan\theta = \frac{3}{2}$ $\boxed{\surd}$

$\theta \doteqdot 56^\circ$ (to nearest degree) $\boxed{\surd}$

QUESTION TWO

(a) (i)



(ii) $d = \sqrt{25+9}$
$= \sqrt{34}$ units. ☑

(iii) gradient $= -\frac{3}{5}$ ☑
so $y - 2 = -\frac{3}{5}(x+1)$
$3x + 5y - 7 = 0.$ ☑

(iv) $p = \left|\dfrac{-6-20-7}{\sqrt{9+25}}\right|$ ☑
$= \dfrac{33}{\sqrt{34}}$ units. ☑
Area $= \frac{1}{2} \times \frac{33}{\sqrt{34}} \times \sqrt{34}$ ☑
$= 33$ units^2. ☑

(v) Midpoint $M = (-\frac{3}{2}, -1)$. ☑
From the diagram, $D = (-7, -1)$ since the diagonals bisect eah other. ☑
The reason must be given.
The alternative is to use the midpoint formula again.

(b) (i) Domain is $-2 < x < 2$ ☑☑

(ii) Range is $y > \frac{1}{2}$ ☑

QUESTION THREE

(a) (i)
$$y = x^2 - \frac{1}{x^2}$$
$$= x^2 - x^{-2}$$
$$\frac{dy}{dx} = 2x + 2x^{-3}$$
$$= 2x + \frac{2}{x^3}.$$ $\boxed{\surd\surd\ (-1 \text{ each error, accept negative index})}$

(ii)
$$y = x^2 e^x$$
$$\frac{dy}{dx} = 2xe^x + x^2e^x.$$ $\boxed{\surd}$

(iii)
$$y = \frac{\log_e x}{x}$$
$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log_e x}{x^2}$$
$$= \frac{1 - \log_e x}{x^2}.$$ $\boxed{\surd\surd\ (-1 \text{ each error})}$

(b)
$$y = ax^3 + bx + 4$$
so $-2 = a + b + 4$

$a + b = -6.$ $\boxed{\surd}$ (1)

Also $\dfrac{dy}{dx} = 3ax^2 + b$

and $\dfrac{dy}{dx} = 0$ at $x = 1$

so $3a + b = 0.$ $\boxed{\surd}$ (2)

(2) − (1) $2a = 6$

so $a = 3$

and $b = -9.$ $\boxed{\surd}$

(c) (i)
$$\int \frac{dx}{(x-4)^2} = \int (x-4)^{-2}\,dx$$ $\boxed{\surd}$
$$\frac{-1}{(x-4)} + \text{c, (where c is a constant)}$$ $\boxed{\surd}$

(ii)
$$\int_5^{e+4} \frac{dx}{x-4} = \Big[\log_e(x-4)\Big]_5^{e+4}$$ $\boxed{\surd}$
$$= \log_e e - \log_e 1$$
$$= 1$$ $\boxed{\surd}$

QUESTION FOUR

(a) (i) $$\frac{y-5}{x-2} \times \frac{y+3}{x} = -1 \quad \boxed{\surd}$$

$$x(x-2) + (y-5)(y+3) = 0$$
$$x^2 - 2x + y^2 - 2y - 15 = 0$$
$$x^2 + y^2 - 2x - 2y - 15 = 0 \quad \boxed{\surd}$$

(ii) $x^2 - 2x + 1 + y^2 - 2y + 1 = 17$

$$(x-1)^2 + (y-1)^2 = 17.$$

The centre is $(1, 1)$ and the radius is $\sqrt{17}$ units. $\boxed{\surd\surd}$

(b) (i)

x	2	3	4	5	6
y	−1·386	−0·863	0	1·116	2.433

$\boxed{\surd}$

Area $\doteqdot \left|\frac{4-2}{6}(-1{\cdot}386 + 4 \times -0{\cdot}863)\right| + \frac{6-4}{6}(4 \times 1{\cdot}116 + 2{\cdot}433)$ $\boxed{\surd\surd}$

$\doteqdot 3{\cdot}91$ square units (to 2 decimal places) $\boxed{\surd}$

(c) $$y = 1 + \cos 2x$$

$$\frac{dy}{dx} = -2\sin 2x \quad \boxed{\surd}$$

$$\text{gradient} = -2\sin\tfrac{\pi}{2} \quad \boxed{\surd}$$

$$y - 1 = -2(x - \tfrac{\pi}{4}) \quad \boxed{\surd}$$

$$y - 1 = -2x + \tfrac{\pi}{2}$$

$$4x + 2y - 2 - \pi = 0 \quad \boxed{\surd}$$

QUESTION FIVE

(a) (i) $T_n = 9 - 2n$

$T_1 = 7,\ T_2 = 5,\ T_3 = 3,\ T_4 = 1,\ T_5 = -1,$

so $a = 7$ $\boxed{\surd}$

and $d = -2.$ $\boxed{\surd}$

(ii) $$S_n = \tfrac{n}{2}[2a + (n-1)d]$$
$$= \tfrac{n}{2}[14 + (n-1) \times -2)$$
$$= \tfrac{n}{2}(16 - 2n)$$
$$= n(8 - n)$$
$$= 8n - n^2. \quad \boxed{\surd}$$

(iii)
$$\begin{aligned} \text{Put } S_n &< -945 \\ 8n - n^2 &< -945 \quad \boxed{\surd} \\ n^2 - 8n - 945 &> 0 \\ (n-35)(n+27) &> 0 \end{aligned}$$

so $n < -27$ or $n > 35$. $\boxed{\surd}$

Since $n > 0$ the least number of terms required is 36. $\boxed{\surd}$

(b) (i) $f(x) = mx^2 - 4mx - m + 15$

$$\begin{aligned} f(3) &= 0 \\ 0 &= 9m - 12m - m + 15 \\ 4m &= 15 \\ m &= \tfrac{15}{4}. \quad \boxed{\surd} \end{aligned}$$

(ii) We require $m > 0$ and $\Delta < 0$, $\boxed{\surd}$

$$\begin{aligned} \Delta &= 16m^2 - 4m(-m+15) \\ &= 16m^2 + 4m^2 - 60m \\ &= 20m^2 - 60m \\ &= 20m(m-3). \end{aligned}$$

Now $20m(m-3) < 0$ $\boxed{\surd}$

so $0 < m < 3$. $\boxed{\surd}$

(iii) Put $-\frac{b}{a} = \frac{c}{a}$ $\boxed{\surd}$

so
$$\begin{aligned} -b &= c \\ 4m &= -m + 15 \\ 5m &= 5 \\ m &= 3. \quad \boxed{\surd} \end{aligned}$$

QUESTION SIX

(a) (i)
$$\begin{aligned} \text{Area } \Delta\text{ABC} &= \tfrac{1}{2} \times 8 \times 8 \\ &= 32\,\text{cm}^2. \\ \text{Area sector} &= \tfrac{1}{2} \times 64 \times \tfrac{\pi}{4} \\ &= \tfrac{32\pi}{4}\,\text{cm}^2. \\ \text{Shaded area} &= 32 - \tfrac{32\pi}{4} \\ &= 8(4-\pi)\,\text{cm}^2. \quad \boxed{\surd\surd\ (-1 \text{ each error})} \end{aligned}$$

(ii) Length $AC = 8\sqrt{2}$ cm.

Length $AP = 8\sqrt{2} - 8$ cm.

Length arc $PB = 8 \times \frac{\pi}{4}$

$= 2\pi$ cm.

Perimeter $= 8\sqrt{2} - 8 + 8 + 2\pi$

$= 8\sqrt{2} + 2\pi$ cm. [√√ (−1 each error)]

(b) (i) $y = \frac{1}{3}x^3 - x^2 + 1$

$$\frac{dy}{dx} = x^2 - 2x$$

Put $x^2 - 2x = 0$ for stationary points

$x(x-2) = 0$

so $x = 0$ or 2.

The stationary points are $(0, 1)$ and $(2, -\frac{1}{3})$. [√]

When $x = 0$ $\dfrac{d^2y}{dx^2} = -2$, so $(0,1)$ is a maximum turning point. [√]

When $x = 2$ $\dfrac{d^2y}{dx^2} = 2$, so $(2, -\frac{1}{3})$ is a minimum turning point. [√]

(ii) $\dfrac{d^2y}{dx^2} = 0$ for points of inflexion

so $2x - 2 = 0$

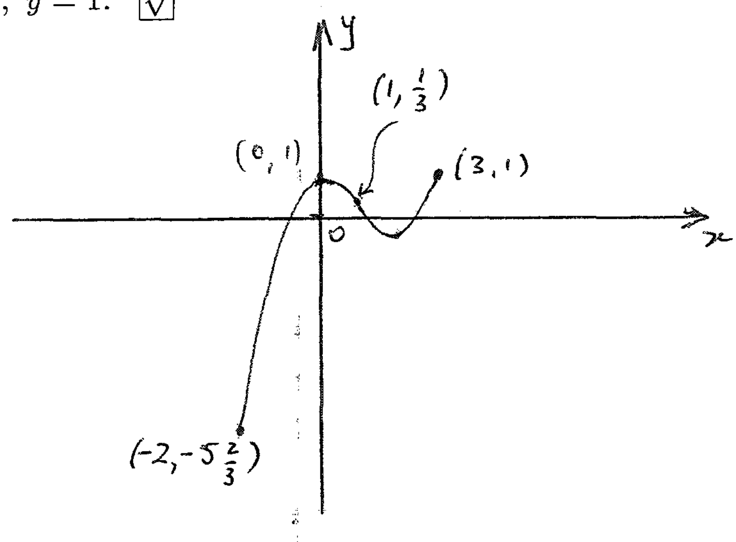
$x = 1$. [√]

x	0	1	2
$\frac{d^2y}{dx^2}$	-2	0	2
	⌢	·	⌣

So the point of inflexion is verified at $(1, \frac{1}{3})$. [√]

(iii) When $x = -2$, $y = -5\frac{2}{3}$,

and when $x = 2$, $y = 1$. [√]



[√√]

QUESTION SEVEN

(a) (i) $x^2 = 4ay$ (1)

$y = a$ (2)

so $x^2 = 4a^2$

$x = -2a$ or $2a$.

So A is $(-2a, a)$ and B is $(2a, a)$. ☑

(ii) $y = \dfrac{x^2}{4a}$,

$$\text{so area} = \int_{-2a}^{2a} (a - \frac{x^2}{4a})\,dx \quad ☑$$

$$= 2\int_{0}^{2a} (a - \frac{x^2}{4a})\,dx$$

$$= 2\left[ax - \frac{x^3}{12a}\right]_0^{2a} \quad ☑$$

$$= 2\left(2a^2 - \frac{8a^2}{12a}\right)$$

$$= \frac{8a^2}{3}\text{ units}^2. \quad ☑$$

(b) (i) $y = 4 - x^2$ (1)

$y = 2 - x$ (2)

(1) − (2) $0 = 2 - x^2 + x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

so $x = -1$ or 2.

At P, $x = -1$ and at Q, $x = 2$. ☑

(ii) $$V = \pi\int_{-1}^{2} (4-x^2)^2 + (2-x)^2\,dx \quad ☑$$

$$= \pi\int_{-1}^{2} (x^4 - 9x^2 + 4x + 12)\,dx$$

$$= \pi\left[\tfrac{x^5}{5} - 3x^3 + 2x^2 + 12x\right]_{-1}^{2} \quad ☑$$

$$= \pi[\tfrac{32}{5} - 24 + 8 + 24] - \pi[-\tfrac{1}{5} + 3 + 2 - 12]$$

$$= \frac{108\pi}{5}\text{ units}^3. \quad ☑$$

(c) (i) $P = P_0e^{kt}$

$9000 = 6000e^{10k}$

$k = \frac{1}{10}\log_e\frac{3}{2}$. ☑

(ii) $30000 = 6000e^{t\left(\frac{1}{10}\log_e \frac{3}{2}\right)}$ ☑

$\log_e 5 = t\left(\frac{1}{10}\log_e \frac{3}{2}\right)$ ☑

$t = \dfrac{10\log_e 5}{\log_e \frac{3}{2}}$

$\doteqdot 37$ years, (to the nearest whole number). ☑

QUESTION EIGHT

(a) (i) $\angle DCE = 52°$ (cointerior angles, $FE \parallel DC$) ☑

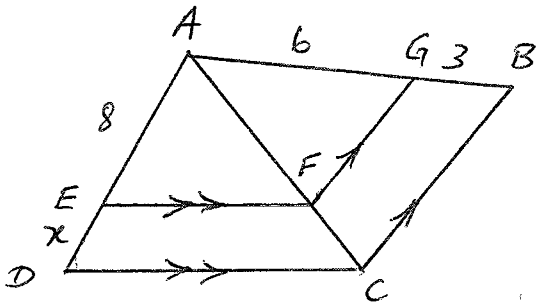
$\angle DCE = 128°$ (cointerior angles, $AB \parallel DC$) ☑

$\angle BCE = \angle BCD - \angle DCE$

$= 128° - 52°$

$= 76°$. ☑

(b) (i)



(ii) $\dfrac{AG}{GB} = \dfrac{AF}{FC}$

$\dfrac{AF}{FC} = \dfrac{AE}{ED}$ (intercept properties on transversals) ☑

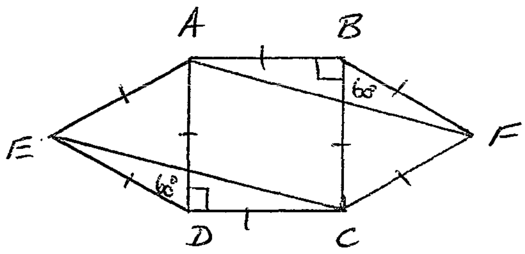
so $\dfrac{AG}{GB} = \dfrac{AE}{ED}$ ☑

$\dfrac{6}{3} = \dfrac{8}{x}$

$x = 4$.

So $ED = 4\,\text{cm}$. ☑

(c) (i)



In square $ABCD$:
$AB = BC = CD = DA$ (equal sides). ☑
In equilateral triangles BFC and ADE:
$BC = CF = FB = AD = AE = ED$ (equal sides of equilateral triangles and $AD = BC$).
So $BF = ED$ and $AB = CD$. ☑

Also $\angle ABF = \angle ABC + \angle CBF$
$= 90° + 60°$ (sum interior angle of a square and an equilateral triangle)
$= 150.°$

Similarly $\angle EDC = 150°$. ☑
Join EC and AF.
In $\triangle ABF$ and $\triangle EDC$,
1. $BF = ED$ from above
2. $AB = CD$ from above
3. $\angle ABF = \angle EDC$ from above

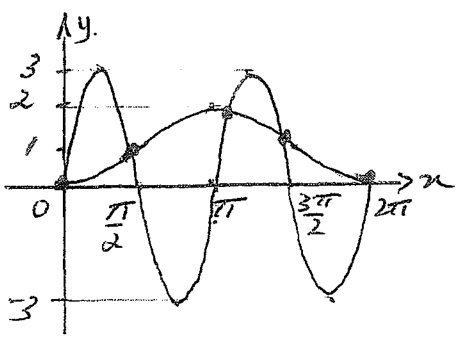
so $\triangle ABF \equiv \triangle EDC$ (SAS). ☑

(ii) $AF = EC$ (matching sides of congruent triangles). ☑

(iii) Now $AF = EC$ and $AE = FC$ so $AFCE$ is a parallelogram (opposite sides equal). ☑

QUESTION NINE

(a) (i)



$\sqrt{}\sqrt{}$ for sine curve

$\sqrt{}\sqrt{}$ for cosine curve

(ii) There are five solutions. ☑

(b) (i) $x = 30t - 150 + 150e^{-0\cdot 2t}$

$v = 30 - 30e^{-0\cdot 2t}$

$v = 30(1 - e^{-0\cdot 2t})$ ☑

(ii) As $t \to \infty$, $e^{-0\cdot 2t} \to 0$ so $v \to 30$ from below.
The velocity does not exceed 30 metres per second. ☑

(iii) Let $v = 15$

so $15 = 30(1 - e^{-0\cdot 2t})$ ☑

$e^{-0\cdot 2t} = \frac{1}{2}$

$t = -\frac{1}{0\cdot 2}\log_e \frac{1}{2}$

$\doteqdot 3\cdot 5$ seconds (to the nearest $0\cdot 1$ second). ☑

$x = 30 \times 3\cdot 5 - 150 + 150e^{-0\cdot 2\times 3\cdot 5}$

$\doteqdot 29$ metres (to the nearest metre). ☑

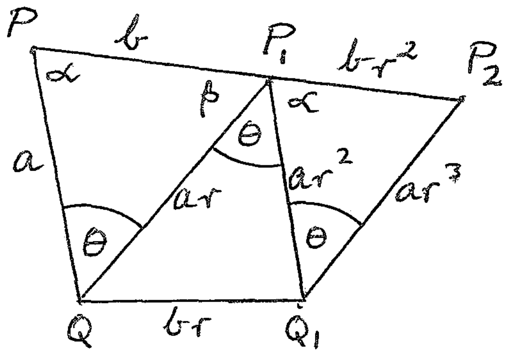
(iv) $\frac{d^2y}{dx^2} = 6e^{-0\cdot 2t}$ ☑

$= 6$ when $t = 0$.

So the initial acceleration is $6\,\mathrm{m/sec}^2$. ☑

QUESTION TEN

(a) (i)



Let $\angle QPP_1 = \alpha$ and $\angle QP_1P = \beta$

so $\theta + \alpha + \beta = 180°$ (angle sum of triangle). ☑

Now $\triangle PP_1Q \,|||\, \triangle P_1P_2Q_1$

so $\angle QPP_1 = \angle Q_1P_1P_2 = \alpha$,

so $\angle PP_1Q + \angle QP_1Q_1 + \angle Q_1P_1P_2 = \alpha + \theta + \beta$

$= 180°$.

So P, P_1 and P_2 are collinear. ☑

(ii)
$$\text{Area } \triangle PQP_1 = \tfrac{1}{2} \times a \times ar \times \sin\theta$$
$$= \tfrac{1}{2}a^2 r \sin\theta. \quad \boxed{\checkmark}$$
$$\text{Area } \triangle QP_1Q_1 = \tfrac{1}{2} \times ar \times ar^2 \times \sin\theta$$
$$= \tfrac{1}{2}a^2 r^3 \sin\theta$$
so
$$\frac{\text{Area } \triangle QP_1Q_1}{\text{Area } \triangle PQP_1} = \frac{\tfrac{1}{2}a^2 r^3 \sin\theta}{\tfrac{1}{2}a^2 r \sin\theta}$$
$$= r^2. \quad \boxed{\checkmark}$$

(b) (i) Length of OP is the limiting sum of the lengths $PP_1,\ P_1P_2,\ P_2P_3,\ \ldots$.

First term $= b$,

common ratio $= r^2$

so
$$OP = \frac{b}{1-r^2}. \quad \boxed{\checkmark}$$

Similarly,
$$OQ = \frac{br}{1-r^2}. \quad \boxed{\checkmark}$$

(ii) Area of $\triangle QOP$ is the limiting sum of the areas of the triangles $PQP_1,\ QP_1Q_1,\ P_1Q_1P_2,\ \ldots$.

First term $= \tfrac{1}{2}ra^2 \sin\theta$,

common ratio $= r^2$.
$$\text{Area } \triangle QOP = \frac{\tfrac{1}{2}ra^2 \sin\theta}{1-r^2}$$
$$= \frac{ra^2 \sin\theta}{2(1-r^2)} \quad \boxed{\checkmark}$$

(iii) Also, area
$$\triangle QOP = \frac{1}{2} \times \frac{b}{1-r^2} \times \frac{br}{1-r^2} \times \sin\alpha$$
$$= \frac{b^2 r \sin\alpha}{2(1-r^2)^2}. \quad \boxed{\checkmark}$$
so
$$\frac{b^2 r \sin\alpha}{2(1-r^2)^2} = \frac{ra^2 \sin\theta}{2(1-r^2)} \quad \boxed{\checkmark}$$
so
$$\frac{\sin\alpha}{\sin\theta} = \frac{a^2(1-r^2)}{b^2} \quad \boxed{\checkmark}$$

(iv) $b = \dfrac{a\sqrt{91}}{10}$

so $b^2 = \dfrac{91a^2}{100}$.

Now $\sin\alpha = \dfrac{a^2(1-r^2)\sin\alpha}{b^2}$

so $\sin\alpha = \dfrac{100(1-0{\cdot}81)\sin 60^\circ}{91}$

$= \dfrac{19\sqrt{3}}{182}$. $\boxed{\surd}$

So $\alpha \doteqdot 10^\circ 25'$ (to the nearest minute) $\boxed{\surd}$

GJ